Rational homotopy – Sullivan models

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Abstract. This chapter is a short introduction to Sullivan models. In particular, we find the Sullivan model of a free loop space and use it to prove the Vigué-Poirrier-Sullivan theorem on the Betti numbers of a free loop space.

In the previous chapter, we have seen the following theorem due to Gromoll and Meyer.

Theorem 0.1. *Let M be a compact simply connected manifold. If the sequence of Betti numbers of the free loop space on M, M^{S^1}, is unbounded then any Riemannian metric on M carries infinitely many non trivial and geometrically distinct closed geodesics.*

In this chapter, using Rational homotopy, we will see exactly when the sequence of Betti numbers of M^{S^1} over a field of caracteristic 0 is bounded (See Theorem 6.1 and its converse Proposition 5.5). This was one of the first major applications of rational homotopy.

Rational homotopy associates to any rational simply connected space, a commutative differential graded algebra. If we restrict to almost free commutative differential graded algebras, that is "Sullivan models", this association is unique.

1 Graded differential algebra

1.1 Definition and elementary properties

All the vector spaces are over $\mathbb{Q}$ (or more generally over a field $\mathbf{k}$ of characteric 0). We will denote by $\mathbb{N}$ the set of non-negative integers.

Definition 1.1. A (non-negatively upper) *graded vector space* V is a family $\{V^n\}_{n\in\mathbb{N}}$ of vector spaces. An element $v \in V_i$ is an element of V of *degree* i. The degree of v is denoted $|v|$. A *differential* d in V is a sequence of linear maps

$d^n : V^n \to V^{n+1}$ such that $d^{n+1} \circ d^n = 0$, for all $n \in \mathbb{N}$. A differential graded vector space or *complex* is a graded vector space equipped with a differential. A morphism of complexes $f : V \overset{\simeq}{\to} W$ is a *quasi-isomorphism* if the induced map in homology $H(f) : H(V) \overset{\cong}{\to} H(W)$ is an isomorphism in all degrees.

Definition 1.2. A *graded algebra* is a graded vector space $A = \{A^n\}_{n\in\mathbb{N}}$, equipped with a multiplication $\mu : A^p \otimes A^q \to A^{p+q}$. The algebra is *commutative* if $ab = (-1)^{|a||b|}ba$ for all a and $b \in A$.

Definition 1.3. A differential graded algebra or *dga* is a graded algebra equipped with a differential $d : A^n \to A^{n+1}$ which is also a *derivation*: this means that for a and $b \in A$

$$d(ab) = (da)b + (-1)^{|a|}a(db).$$

A *cdga* is a commutative dga.

Example 1.4. 1) Let (B, d_B) and (C, d_C) be two cdgas. Then the tensor product $B \otimes C$ equipped with the multiplication

$$(b \otimes c)(b' \otimes c') := (-1)^{|c||b'|}bb' \otimes cc'$$

and the differential

$$d(b \otimes c) = (db) \otimes c + (-1)^{|b|}b \otimes dc.$$

is a cdga. The *tensor product of cdgas* is the sum (or coproduct) in the category of cdgas.

2) More generally, let $f : A \to B$ and $g : A \to C$ be two morphisms of cdgas. Let $B \otimes_A C$ be the quotient of $B \otimes C$ by the sub graded vector spanned by elements of the form $bf(a) \otimes c - b \otimes g(a)c$, $a \in A$, $b \in B$ and $c \in C$. Then $B \otimes_A C$ is a cgda such that the quotient map $B \otimes C \twoheadrightarrow B \otimes_A C$ is a morphism of cdgas. The cdga $B \otimes_A C$ is the pushout of f and g in the category of cdgas:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r] \arrow[drr, bend right] & B \otimes_A C \arrow[dr, dotted, "\exists!"] & \\
 & & D
\end{tikzcd}$$

3) Let V and W be two graded vector spaces. We denote by ΛV the free graded commutative algebra on V.

If $V = \mathbb{Q}v$, i. e. is of dimension 1 and generated by a single element v, then

-ΛV is $E(v) = \mathbb{Q} \oplus \mathbb{Q}v$, the exterior algebra on v if the degree of v is odd and

-ΛV is $\mathbb{Q}[v] = \oplus_{n\in\mathbb{N}}\mathbb{Q}v^n$, the polynomial or symmetric algebra on v if the degree of v is even.

Since Λ is left adjoint to the forgetful functor from the category of commutative graded algebras to the category of graded vector spaces, Λ preserves sums: there is a natural isomorphism of commutative graded algebras $\Lambda(V \oplus W) \cong \Lambda V \otimes \Lambda W$.

Therefore ΛV is the tensor product $E(V^{odd}) \otimes S(V^{even})$ of the exterior algebra on the generators of odd degree and of the polynomial algebra on the generators of even degree.

Definition 1.5. Let $f : A \to B$ be a morphism of commutative graded algebras. Let $d : A \to B$ be a linear map of degree k. By definition, d is a (f, f)-*derivation* if for a and $b \in A$

$$d(ab) = (da)f(b) + (-1)^{k|a|} f(a)(db).$$

Property 1.6 (Universal properties). 1) Let $i_B : B \hookrightarrow B \otimes \Lambda V$, $b \mapsto b \otimes 1$ and $i_V : V \hookrightarrow B \otimes \Lambda V$, $v \mapsto 1 \otimes v$ be the inclusion maps. Let $\varphi : B \to C$ be a morphism of commutative graded algebras. Let $f : V \to C$ be a morphism of graded vector spaces. Then φ and f extend uniquely to a morphism $B \otimes \Lambda V \to C$ of commutative graded algebras such that the following diagram commutes

$$\begin{tikzcd}
B \arrow[r, "\varphi"] \arrow[dr, "i_B"'] & C & V \arrow[l, "f"'] \arrow[dl, "i_V"] \\
& B \otimes \Lambda V \arrow[u, dotted, "\exists!"] &
\end{tikzcd}$$

2) Let $d_B : B \to B$ be a derivation of degree k. Let $d_V : V \to B \otimes \Lambda V$ be a linear map of degree k. Then there is a unique derivation d such that the following diagram commutes.

$$\begin{tikzcd}
B \arrow[r, "i_B"] & B \otimes \Lambda V & V \arrow[l, "d_V"'] \arrow[dl, "i_V"] \\
B \arrow[u, "d_B"] \arrow[r, "i_B"] & B \otimes \Lambda V \arrow[u, dotted, "\exists! d"] &
\end{tikzcd}$$

3) Let $f : \Lambda V \to B$ be a morphism of commutative graded algebras. Let $d_V : V \to B$ be a linear map of degree k. Then there exists a unique (f, f)-derivation d extending d_V:

$$\begin{tikzcd}
V \arrow[r, "d_V"] \arrow[d, "i_V"'] & B \\
\Lambda V \arrow[ur, dotted, "\exists! d"'] &
\end{tikzcd}$$

Proof. 1) Since ΛV is the free commutative graded algebra on V, f can be extended to a morphism of graded algebras $\Lambda V \to C$. Since the tensor product of commutative graded algebras is the sum in the category of commutative graded algebras, we obtain a morphism of commutative graded algebras from $B \otimes \Lambda V$ to C.

2) Since $b \otimes v_1 \ldots v_n$ is the product $(b \otimes 1)(1 \otimes v_1) \ldots (1 \otimes v_n)$, $d(b \otimes v_1 \ldots v_n)$

is given by

$$d_B(b)\otimes v_1\dots v_n+\sum_{i=1}^{n}(-1)^{k(|b|+|v_1|+\dots+|v_{i-1}|)}(b\otimes v_1\dots v_{i-1})(d_V v_i)(1\otimes v_{i+1}\dots v_n)$$

3) Similarly, $d(v_1\dots v_n)$ is given by

$$\sum_{i=1}^{n}(-1)^{k(|v_1|+\dots+|v_{i-1}|)}f(v_1)\dots f(v_{i-1})d_V(v_i)f(v_{i+1})\dots f(v_n)$$

$\square$

1.2 Sullivan models of spheres

Sullivan models of odd spheres S^{2n+1}, $n\geq 0$.

Consider a cdga $A(S^{2n+1})$ whose cohomology is isomorphic as graded algebras to the cohomology of S^{2n+1} with coefficients in $\mathbf{k}$:

$$H^*(A(S^{2n+1}))\cong H^*(S^{2n+1}).$$

When $\mathbf{k}$ is $\mathbb{R}$, you can think of A as the De Rham algebra of forms on S^{2n+1}. There exists a cycle v of degree $2n+1$ in $A(S^{2n+1})$ such that

$$H^*(A(S^{2n+1}))=\Lambda[v].$$

The inclusion of complexes $(\mathbf{k}v,0)\hookrightarrow A(S^{2n+1})$ extends to a unique morphism of cdgas $m:(\Lambda v,0)\to A(S^{2n+1})$(Property 1.6):

$$\begin{tikzcd}
(\mathbf{k}v,0) \arrow[r] \arrow[d] & A(S^{2n+1}) \\
(\Lambda v,0) \arrow[ur, dotted, "\exists! m"'] &
\end{tikzcd}$$

The induced morphism in homology $H(m)$ is an isomorphism. We say that $m:(\Lambda v,0)\overset{\simeq}{\twoheadrightarrow} A(S^{2n+1})$ is a Sullivan model of S^{2n+1}

Sullivan models of even spheres S^{2n}, $n\geq 1$.

Exactly as above, we construct a morphism of cdga $m_1:(\Lambda v,0)\to A(S^{2n})$. But now, $H(m_1)$ is not an isomorphism:

$H(m_1)(v)=[v]$. Therefore $H(m_1)(v^2)=[v^2]=[v]^2=0$. Since $[v^2]=0$ in $H^*(A(S^{2n}))$, there exists an element $\psi\in A(S^{2n})$ of degree $4n-1$ such that $d\psi=v^2$.

Let w denote another element of degree $4n-1$. The morphism of graded vector spaces $\mathbf{k}v\oplus\mathbf{k}w\hookrightarrow A(S^{2n})$, mapping v to v and w to ψ extends to a unique morphism of commutative graded algebras $m:\Lambda(v,w)\to A(S^{2n})$ (1) of

Property 1.6):

$$\begin{tikzcd} \mathbf{k}v \oplus \mathbf{k}w \arrow[r] \arrow[d] & A(S^{2n}) \\ \Lambda(v,w) \arrow[ur, dotted, "\exists! m"'] & \end{tikzcd}$$

The linear map of degree $+1$, $d_V : V := \mathbf{k}v \oplus \mathbf{k}w \to \Lambda(v,w)$ mapping v to 0 and w to v^2 extends to a unique derivation $d : \Lambda(v,w) \to \Lambda(v,w)$ (2) of Property 1.6).

$$\begin{tikzcd} \mathbf{k}v \oplus \mathbf{k}w \arrow[r, "d_V"] \arrow[d] & \Lambda(v,w) \\ \Lambda(v,w) \arrow[ur, dotted, "\exists! d"'] & \end{tikzcd}$$

Since d is a derivation of odd degree, $d \circ d$ (which is equal to $1/2[d,d]$) is again a derivation. The following diagram commutes

$$\begin{tikzcd} V \arrow[r, "d_V"] \arrow[d] & \Lambda V \arrow[r, "d"] & \Lambda V \\ \Lambda V \arrow[ur, "d"] \arrow[urr, "d \circ d"'] & & \end{tikzcd}$$

Since the composite $d \circ d_V$ is null, by unicity (2) of Property 1.6), the derivation $d \circ d$ is also null. Therefore $(\Lambda V, d)$ is a cdga. This is the general method to check that $d \circ d = 0$.

Denote by d_A the differential on $A(S^{2n})$. Let's check now that $d_A \circ m = m \circ d$. Since d_A and d are both (id, id)-derivations, $d_A \circ m$ and $m \circ d$ are both (m,m)-derivations.

Since $d_A(m(v)) = d_A(v) = 0 = m(0) = m(d(v))$ and $d_A(m(w)) = d_A(\psi) = v^2 = m(v^2) = m(d(w))$, $d_A \circ m$ and $m \circ d$ coincide on V. Therefore by unicity (3) of Property 1.6), $d_A \circ m = m \circ d$. Again, this method is general. So finally, we have proved that m is a morphism of cdgas. Now we prove that $H(m)$ is an isomorphism, by checking that $H(m)$ sends a basis to a basis.

2 Sullivan models

2.1 Definitions

Let V be a graded vector space. Denote by $V^+ = V^{\geq 1}$ the sub graded vector space of V formed by the elements of V of positive degrees: $V = V^0 \oplus V^+$.

Definition 2.1. A *relative Sullivan model* (or *cofibration* in the category of cdgas) is a morphism of cdgas of the form

$$(B, d_B) \hookrightarrow (B \otimes \Lambda V, d), b \mapsto b \otimes 1$$

where

- $H^0(B) \cong \mathbf{k}$,
- $V = V^{\geq 1}$,
- and V is the direct sum of graded vector spaces $V(k)$:

$$\forall n, V^n = \bigoplus_{k\in\mathbb{N}} V(k)^n$$

such that $d : V(0) \to B \otimes \mathbf{k}$ and $d : V(k) \to B \otimes \Lambda(V(< k))$. Here $V(< k)$ denotes the direct sum $V(0) \oplus \cdots \oplus V(k-1)$.

Let $k \in \mathbb{N}$. Denote by $\Lambda^k V$ the sub graded vector space of ΛV generated by elements of the form $v_1 \wedge \cdots \wedge v_k$, $v_i \in V$. Elements of $\Lambda^k V$ have by definition *wordlength* k. For example $\Lambda V = \mathbf{k} \oplus V \oplus \Lambda^{\geq 2} V$.

Definition 2.2. A relative Sullivan model $(B, d_B) \hookrightarrow (B \otimes \Lambda V, d)$ is *minimal* if $d : V \to B^+ \otimes \Lambda V + B \otimes \Lambda^{\geq 2} V$. A *(minimal) Sullivan model* is a (minimal) relative Sullivan model of the form $(B, d_B) = (\mathbf{k}, 0) \hookrightarrow (\Lambda V, d)$.

Example 2.3. [5, end of the proof of Lemma 23.1] Let $(\Lambda V, d)$ be cdga such that $V = V^{\geq 2}$. Then $(\Lambda V, d)$ is a Sullivan model.

proof assuming the minimality condition. [5, p. 144] Suppose that $d : V \to \Lambda^{\geq 2} V$. In this case, the $V(k)$ are easy to define: let $V(k) := V^k$ for $k \in N$. Let $v \in V^k$. By the minimality condition, dv is equal to a sum $\sum_i x_i y_i$ where the non trivial elements x_i and y_i are both of positive length and therefore both of degre ≥ 2. Since $|x_i| + |y_i| = |dv| = k + 1$, both x_i and y_i are of degree less than k. Therefore dv belongs to $\Lambda(V^{<k}) = \Lambda(V(< k))$. □

Property 2.4. The composite of relative Sullivan models is again a Sullivan relative model.

Definition 2.5. Let C be a cdga. A (minimal) *Sullivan model of* C is a (minimal) Sullivan model $(\Lambda V, d)$ such that there exists a quasi-isomorphism of cdgas $(\Lambda V, d) \overset{\simeq}{\twoheadrightarrow} C$.

Let $\varphi : B \to C$ be a morphism of cdgas. A (minimal) *relative Sullivan model of* φ is a (minimal) relative Sullivan model $(B, d_B) \hookrightarrow (B \otimes \Lambda V, d)$ such that φ can be decomposed as the composite of the relative Sullivan model and of a quasi-isomorphism of cdgas:

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[dr] & C \\ & B \otimes \Lambda V \arrow[u, "\simeq"'] \end{tikzcd}$$

Theorem 2.6. *Any morphism $\varphi : B \to C$ of cdgas admits a minimal relative Sullivan model if $H^0(B) \cong \mathbf{k}$, $H^0(\varphi)$ is an isomorphism and $H^1(\varphi)$ is injective.*

This theorem is proved in general by Proposition 14.3 and Theorem 14.9 of [5]. But in practice, if $H^1(\varphi)$ is an isomorphism, we construct a minimal relative Sullivan model, by induction on degrees as in Proposition 12.2. of [5].

2.2 An example of relative Sullivan model

Consider the minimal Sullivan model of an odd sphere found in section 1.2

$$(\Lambda v, 0) \overset{\simeq}{\to} A(S^{2n+1}).$$

Assume that $n \geq 1$. Consider the multiplication of Λv: the morphism of cdgas

$$\mu : (\Lambda v_1, 0) \otimes (\Lambda v_2, 0) \to (\Lambda v, 0), v_1 \mapsto v, v_2 \mapsto v.$$

Recall that v, v_1 and v_2 are of degree $2n+1$.
Denote by sv an element of degree $|sv| = |s| + |v| = -1 + |v|$. The operator s of degre -1 is called the *suspension*.

We construct now a minimal relative Sullivan model of μ. Define $d(sv) = v_2 - v_1$. Let $m : \Lambda(v_1, v_2, sv), d \to (\Lambda v, 0)$ be the unique morphism of cdgas extending μ such that $m(sv) = 0$.

$$\begin{tikzcd}
(\Lambda v_1, 0) \otimes (\Lambda v_2, 0) \arrow[r, "\mu"] \arrow[dr] & (\Lambda v, 0) \\
& \Lambda(v_1, v_2, sv, d) \arrow[u, "m"']
\end{tikzcd}$$

Definition 2.7. Let A be a differential graded algebra such that $A^0 = \mathbf{k}$. The complex of indecomposables of A, denoted $Q(A)$, is the quotient $A^+/\mu(A^+ \otimes A^+)$.

The complex of indecomposables of $(\Lambda v, 0)$, $Q((\Lambda v, 0))$, is $(\mathbf{k}v, 0)$ while

$$Q(\Lambda(v_1, v_2, sv, d)) = (\mathbf{k}v_1 \oplus \mathbf{k}v_2 \oplus \mathbf{k}sv, d(sv) = v_2 - v_1).$$

The morphism of complexes $Q(m) : (\mathbf{k}v_1 \oplus \mathbf{k}v_2 \oplus \mathbf{k}sv, d(sv) = v_2 - v_1) \to (\mathbf{k}v, 0)$ map v_1 to v, v_2 to v and sv to 0. It is easy to check that $Q(m)$ is a quasi-isomorphism of complexes.

By Proposition 14.13 of [5], since m is a morphism of cdgas between Sullivan model, $Q(m)$ is a quasi-isomorphim of if and only if m is a quasi-isomorphism.

So we have proved that m is a quasi-isomorphism and therefore

$$(\Lambda v_1, 0) \otimes (\Lambda v_2, 0) \hookrightarrow \Lambda(v_1, v_2, sv, d)$$

is a minimal relative Sullivan model of μ. Consider the following commutative diagram of cdgas where the square is a pushout

$$\begin{tikzcd}
 & & \Lambda v, 0 \\
\Lambda(v_1, v_2), 0 \arrow[r] \arrow[urr, "\mu"] \arrow[d, "\mu"'] & \Lambda(v_1, v_2, sv), d \arrow[ur, "\simeq", "m"'] \arrow[d] & \\
\Lambda v, 0 \arrow[r] & \Lambda v, 0 \otimes_{\Lambda(v_1,v_2),0} \Lambda(v_1, v_2, sv), d &
\end{tikzcd}$$

It is easy to check that the cdga $\Lambda v, 0 \otimes_{\Lambda(v_1,v_2),0} \Lambda(v_1, v_2, sv), d$ is isomorphic to $\Lambda(v, sv), 0$. As we will explain later, we have computed in fact, the minimal Sullivan model $\Lambda(v, sv), 0$ of the free loop space $(S^{2n+1})^{S^1}$. In particular, the cohomology algebra $H^*((S^{2n+1})^{S^1}; \mathbf{k})$ is isomorphic to $\Lambda(v, sv)$. We can deduce easily that for $p \in \mathbb{N}$, $\dim H^p((S^{2n+1})^{S^1}) \leq 1$. So we have shown that the sequence of Betti numbers of the free loop space on odd dimensional spheres is bounded.

2.3 The relative Sullivan model of the multiplication

Proposition 2.8. *[6, Example 2.48] Let $(\Lambda V, d)$ be a relative minimal Sullivan model with $V = V^{\geq 2}$ (concentrated in degrees ≥ 2). Then the multiplication $\mu : (\Lambda V, d) \otimes (\Lambda V, d) \twoheadrightarrow (\Lambda V, d)$ admits a minimal relative Sullivan model of the form $(\Lambda V \otimes \Lambda V \otimes \Lambda sV, D)$.*

Constructive proof. We proceed by induction on $n \in \mathbb{N}^*$ to construct quasi-isomorphisms of cdgas $\varphi_n : (\Lambda V^{\leq n} \otimes \Lambda V^{\leq n} \otimes \Lambda sV^{\leq n}, D) \overset{\simeq}{\twoheadrightarrow} (\Lambda V^{\leq n}, d)$ extending the multiplication on $\Lambda V^{\leq n}$.

Suppose that φ_n is constructed. We now define φ_{n+1} extending φ_n and μ, the multiplication on ΛV. Let $v \in V^{n+1}$. Then $d(v) \in \Lambda^{\geq 2}(V^{\leq n})$ and $\varphi_n(dv \otimes 1 \otimes 1 - 1 \otimes dv \otimes 1) = 0$. Since φ_n is a surjective quasi-isomorphism, by the long exact sequence associated to a short exact sequence of complexes, $\operatorname{Ker} \varphi_n$ is acyclic. Therefore since $dv \otimes 1 \otimes 1 - 1 \otimes dv \otimes 1$ is a cycle, there exists an element γ of degree $n+1$ of $\Lambda V^{\leq n} \otimes \Lambda V^{\leq n} \otimes \Lambda sV^{\leq n}$ such that $D(\gamma) = dv \otimes 1 \otimes 1 - 1 \otimes dv \otimes 1$ and $\varphi_n(\gamma) = 0$. For degree reasons, γ is decomposable, i. e. has wordlength ≥ 2. We define $D(1 \otimes 1 \otimes sv) = v \otimes 1 \otimes 1 - 1 \otimes v \otimes 1 - \gamma$ and $\varphi_{n+1}(1 \otimes 1 \otimes sv) = 0$. Since $D \circ D(1 \otimes 1 \otimes sv) = 0$ and $d \circ \varphi_{n+1}(1 \otimes 1 \otimes sv) = \varphi_{n+1} \circ d(1 \otimes 1 \otimes sv)$, by Property 1.6, the derivation D is a differential on $\Lambda V^{\leq n+1} \otimes \Lambda V^{\leq n+1} \otimes \Lambda sV^{\leq n+1}$ and the morphism of graded algebras φ_{n+1} is a morphism of complexes.

The complex of indecomposables of $(\Lambda V^{\leq n+1} \otimes \Lambda V^{\leq n+1} \otimes \Lambda sV^{\leq n+1}, D)$,

$$Q((\Lambda V^{\leq n+1} \otimes \Lambda V^{\leq n+1} \otimes \Lambda sV^{\leq n+1}, D)$$

is $(V^{\leq n+1} \oplus V^{\leq n+1} \oplus sV^{\leq n+1}, d)$ with differential d given by $d(v' \oplus v" \oplus sv) = v \oplus -v \oplus 0$ for v', $v"$ and $v \in V^{\leq n+1}$. Therefore it is easy to check that $Q(\varphi_{n+1})$ is a quasi-isomorphism. So by Proposition 14.13 of [5], φ_{n+1} is a quasi-isomorphism. Since γ is of degree $n+1$ and $sV^{\leq n}$ is of degree $< n$, this relative Sullivan model is minimal. We now define $\varphi : (\Lambda V \otimes \Lambda V \otimes \Lambda sV, D) \twoheadrightarrow (\Lambda V, d)$ as

$$\varinjlim \varphi_n = \bigcup_{n\in\mathbb{N}} \varphi_n : \bigcup_{n\in\mathbb{N}} \left(\Lambda V^{\leq n} \otimes \Lambda V^{\leq n} \otimes \Lambda sV^{\leq n}\right) \to \bigcup_{n\in\mathbb{N}} \Lambda V^{\leq n}.$$

Since homology commutes with direct limits in the category of complexes [14, Chap 4, Sect 2, Theorem 7], $H(\varphi) = \varinjlim H(\varphi_n)$ is an isomorphism. □

3 Rational homotopy theory

Let X be a topological space. Denote by $S^*(X)$ the singular cochains of X with coefficients in $\mathbf{k}$. The dga $S^*(X)$ is almost never commutative. Nevertheless, Sullivan, inspired by Quillen proved the following theorem.

Theorem 3.1. *[5, Corollary 10.10] For any topological space X, there exists two natural quasi-isomorphisms of dgas*

$$S^*(X) \overset{\simeq}{\to} D(X) \overset{\simeq}{\leftarrow} A_{PL}(X)$$

where $A_{PL}(X)$ is commutative.

Remark 3.2. This cdga $A_{PL}(X)$ is called the algebra of *polynomial differential forms*. If $\mathbf{k} = \mathbb{R}$ and X is a smooth manifold M, you can think that $A_{PL}(M)$ is the De Rham algebra of differential forms on M, $A_{DR}(M)$ [5, Theorem 11.4].

Definition 3.3. [6, Definition 2.34] Two topological spaces X and Y have the same *rational homotopy type* if there exists a finite sequence of continuous applications

$$X \overset{f_0}{\to} Y_1 \overset{f_1}{\leftarrow} Y_2 \dots Y_{n-1} \overset{f_{n-1}}{\leftarrow} Y_n \overset{f_n}{\to} Y$$

such that the induced maps in rational cohomology

$$H^*(X;\mathbb{Q}) \overset{H^*(f_0)}{\leftarrow} H^*(Y_1;\mathbb{Q}) \overset{H^*(f_1)}{\to} H^*(Y_2;\mathbb{Q}) \dots H^*(Y_{n-1}1;\mathbb{Q})$$
$$\overset{H^*(f_{n-1})}{\to} H^*(Y_n;\mathbb{Q}) \overset{H^*(f_n)}{\leftarrow} H^*(Y;\mathbb{Q})$$

are all isomorphisms.

Theorem 3.4. *Let X be a path connected topological space.*

1) (Unicity of minimal Sullivan models [5, Corollary p. 191]) Two minimal Sullivan models of $A_{PL}(X)$ are isomorphic.

2) Suppose that X is simply connected and $\forall n \in \mathbb{N}$, $H_n(X;\mathbf{k})$ is finite dimensional. Let $(\Lambda V, d)$ be a minimal Sullivan model of X. Then [5, Theorem 15.11] for all $n \in \mathbb{N}$, V^n is isomorphic to $Hom_{\mathbf{k}}(\pi_n(X) \otimes_{\mathbb{Z}} \mathbf{k}, \mathbf{k}) \cong Hom_{\mathbb{Z}}(\pi_n(X), \mathbf{k})$. In particular [5, Remark 1 p.208], Dimension V^n = Dimension $\pi_n(X) \otimes_{\mathbb{Z}} \mathbf{k} < \infty$.

Remark 3.5. The isomorphim of graded vector spaces between V and $\mathrm{Hom}_{\mathbf{k}}(\pi_*(X)\otimes_{\mathbb{Z}} \mathbf{k}, \mathbf{k})$ is natural in some sense [6, p. 75-6] with respect to maps $f : X \to Y$. The isomorphism behaves well also with respect to the long exact sequence associated to a (Serre) fibration ([5, Proposition 15.13] or [6, Proposition 2.65]).

Theorem 3.6. *[6, Proposition 2.35][5, p. 139] Let X and Y be two simply connected topological spaces such that $H^n(X;\mathbb{Q})$ and $H^n(Y;\mathbb{Q})$ are finite dimensional for all $n \in \mathbb{N}$. Let $(\Lambda V, d)$ be a minimal Sullivan model of X and let $(\Lambda W, d)$ be a minimal Sullivan model of Y. Then X and Y have the same rational homotopy type if and only if $(\Lambda V, d)$ is isomorphic to $(\Lambda W, d)$ as cdgas.*

4 Sullivan model of a pullback

4.1 Sullivan model of a product

Let X and Y be two topological spaces. Let $p_1 : X \times Y \twoheadrightarrow Y$ and $p_2 : X \times Y \twoheadrightarrow X$ be the projection maps. Let m be the unique morphism of cdgas given by the universal property of the tensor product (Example 1.4 1))

$$\begin{tikzcd}
& A_{PL}(Y) \arrow[d] \arrow[ddr, bend left, "A_{PL}(p_2)"] & \\
A_{PL}(X) \arrow[r] \arrow[drr, bend right, "A_{PL}(p_1)"'] & A_{PL}(X)\otimes A_{PL}(Y) \arrow[dr, dotted, "\exists! m"] & \\
& & A_{PL}(X\times Y).
\end{tikzcd}$$

Assume that $H^*(X;\mathbf{k})$ or $H^*(Y;\mathbf{k})$ is finite dimensional in all degrees. Then [5, Example 2, p. 142-3] m is a quasi-isomorphism. Let $m_X : \Lambda V \overset{\simeq}{\to} A_{PL}(X)$ be a Sullivan model of X. Let $m_Y : \Lambda W \overset{\simeq}{\to} A_{PL}(Y)$ be a Sullivan model of Y. Then by Künneth theorem, the composite

$$\Lambda V \otimes \Lambda W \overset{m_X \otimes m_Y}{\to} A_{PL}(X) \otimes A_{PL}(Y) \overset{m}{\to} A_{PL}(X \times Y)$$

is a quasi-isomorphism of cdgas. Therefore we have proved that "the Sullivan model of a product is the tensor product of the Sullivan models".

4.2 the model of the diagonal

Let X be a topological space such that $H^*(X)$ is finite dimensional in all degrees. Denote by $\Delta : X \to X \times X$, $x \mapsto (x, x)$ the diagonal map of X. Using the previous paragraph, since $A_{PL}(p_1 \circ \Delta) = A_{PL}(p_2 \circ \Delta) = A_{PL}(\mathrm{id}) = \mathrm{id}$, we have the commutative diagram of cdgas.

$$\begin{tikzcd}
A_{PL}(X) \arrow[r] \arrow[dr, "A_{PL}(p_1)"'] \arrow[ddr, bend right, "\mathrm{id}"'] & A_{PL}(X)\otimes A_{PL}(X) \arrow[d, "m", "\simeq"'] & A_{PL}(X) \arrow[l] \arrow[dl, "A_{PL}(p_2)"] \arrow[ddl, bend left, "\mathrm{id}"] \\
& A_{PL}(X\times X) \arrow[d, "A_{PL}(\Delta)"] & \\
& A_{PL}(X) &
\end{tikzcd}$$

Therefore the composite $A_{PL}(X)\otimes A_{PL}(X) \overset{m}{\to} A_{PL}(X\times X) \overset{A_{PL}(\Delta)}{\to} A_{PL}(X)$ coincides with the multiplication $\mu : A_{PL}(X)\otimes A_{PL}(X) \to A_{PL}(X)$. Therefore the following diagram of cdgas commutes

$$\begin{tikzcd}
A_{PL}(X) & A_{PL}(X\times X) \arrow[l, "A_{PL}(\Delta)"'] \\
& A_{PL}(X)\otimes A_{PL}(X) \arrow[u, "m"', "\simeq"] \arrow[ul, "\mu"'] \\
\Lambda V \arrow[uu, "m_X", "\simeq"'] & \Lambda V\otimes\Lambda V \arrow[l, "\mu"] \arrow[u, "m_X\otimes m_X"', "\simeq"]
\end{tikzcd}$$

Here $m_X : \Lambda V \overset{\simeq}{\to} A_{PL}(X)$ denotes a Sullivan model of X. Therefore we have proved that "the morphism modelling the diagonal map is the multiplication of the Sullivan model".

4.3 Sullivan model of a fibre product

Consider a pullback square in the category of topological spaces

$$\begin{tikzcd}
P \arrow[r, "g"] \arrow[d, "q"'] & E \arrow[d, "p"] \\
X \arrow[r, "f"] & B
\end{tikzcd}$$

where

- $p : E \to B$ is a (Serre) fibration between two topological spaces,
- for every $i \in \mathbb{N}$, $H^i(X)$ and $H^i(B)$ are finite dimensional,
- the topological spaces X and E are path-connected and B is simply-connected.

Since p is a (Serre) fibration, the pullback map q is also a (Serre) fibration. Let $A_{PL}(B)\otimes\Lambda V$ be a relative Sullivan model of $A(p)$. Consider the corresponding commutative diagram of cdgas

$$\begin{tikzcd}
& A_{PL}(B) \arrow[r, "A_{PL}(f)"] \arrow[d] \arrow[dl, bend right, "A_{PL}(p)"'] & A_{PL}(X) \arrow[d] \arrow[dr, bend left, "A_{PL}(q)"] & \\
& A_{PL}(B)\otimes\Lambda V \arrow[r] \arrow[dl, "m", "\simeq"'] & A_{PL}(X)\otimes_{A_{PL}(B)} A_{PL}(B)\otimes\Lambda V \arrow[dr, dotted, "\exists ! m'"] & \\
A_{PL}(E) \arrow[rrr, "A_{PL}(g)"] & & & A_{PL}(P)
\end{tikzcd}$$

where the rectangle is a pushout and m' is given by the universal property. Explicitly, for $x \in A_{PL}(X)$ and $e \in A_{PL}(B)\otimes\Lambda V$, $m'(x\otimes e)$ is the product of $A_{PL}(q)(x)$ and $A_{PL}(g)\circ m(e)$.

Since $A_{PL}(B) \hookrightarrow A_{PL}(B)\otimes\Lambda V$ is a relative Sullivan model, the inclusion obtained via pullback $A_{PL}(X) \hookrightarrow A_{PL}(X)\otimes_{A_{PL}(B)} (A_{PL}(B)\otimes\Lambda V, d) \cong (A_{PL}(X)\otimes \Lambda V, d)$ is also a relative Sullivan model (minimal if $A_{PL}(B) \hookrightarrow A_{PL}(B)\otimes\Lambda V$ is minimal).

By [5, Proposition 15.8] (or for weaker hypothesis [6, Theorem 2.70]),

Theorem 4.1. *The morphism of cdgas m' is a quasi-isomorphism.*

We can summarize this theorem by saying that: "The push-out of a (minimal) relative Sullivan model of a fibration is a (minimal) relative Sullivan model of the pullback of the fibration."

Idea of the proof. Since by [5, Lemma 14.1], $A_{PL}(B)\otimes\Lambda V$ is a "semi-free" resolution of $A_{PL}(E)$ as left $A_{PL}(B)$-modules, by definition of the differential torsion product,

$$\mathrm{Tor}^{A_{PL}(B)}(A_{PL}(X), A_{PL}(E)) := H(A_{PL}(X)\otimes_{A_{PL}(B)} (A_{PL}(B)\otimes\Lambda V).$$

By Theorem 3.1 and naturality, we have an isomorphim of graded vector spaces

$$\mathrm{Tor}^{A_{PL}(B)}(A_{PL}(X), A_{PL}(E)) \cong \mathrm{Tor}^{S^*(B)}(S^*(X), S^*(E)).$$

The Eilenberg-Moore formula gives an isomorphism of graded vector spaces

$$\mathrm{Tor}^{S^*(B)}(S^*(X), S^*(E)) \cong H^*(P).$$

We claimed that the resulting isomorphism between the homology of $A_{PL}(X)\otimes_{A_{PL}(B)} (A_{PL}(B)\otimes\Lambda V)$ and $H^*(P)$ can be identified with $H(m)$. Therefore m is a quasi-isomorphism. ☐

Instead of working with A_{PL}, we prefer usually to work at the level of Sullivan models. Let $m_B : \Lambda B \overset{\simeq}{\twoheadrightarrow} A_{PL}(B)$ be a Sullivan model of B. Let $m_X : \Lambda X \overset{\simeq}{\twoheadrightarrow} A_{PL}(X)$ be a Sullivan model of X. Let φ be a morphism of cdgas such the following diagram commutes exactly

$$\begin{tikzcd}
A_{PL}(B) \arrow[r, "A_{PL}(f)"] & A_{PL}(X) \\
\Lambda B \arrow[u, "m_B", "\simeq"'] \arrow[r, "\varphi"] & \Lambda X \arrow[u, "m_X", "\simeq"']
\end{tikzcd}$$

Let $\Lambda B \hookrightarrow \Lambda B \otimes \Lambda V$ be a relative Sullivan model of $A_{PL}(p) \circ m_B$. Consider the corresponding commutative diagram of cdgas

$$\begin{tikzcd}
A_{PL}(B) \arrow[dd, "A_{PL}(p)"'] & \Lambda B \arrow[l, "m_B"', "\simeq"] \arrow[r, "\varphi"] \arrow[d] & \Lambda X \arrow[r, "m_X", "\simeq"'] \arrow[d] & A_{PL}(X) \arrow[dd, "A_{PL}(q)"] \\
 & \Lambda B \otimes \Lambda V \arrow[r] \arrow[dl, "m"', "\simeq"] & \Lambda X \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V) \arrow[dr, dotted, "\exists ! m'"] & \\
A_{PL}(E) \arrow[rrr, "A_{PL}(g)"] & & & A_{PL}(P)
\end{tikzcd} \tag{1}$$

where the rectangle is a pushout and m' is given by the universal property. Then again, $\Lambda X \hookrightarrow \Lambda X \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V)$ is a relative Sullivan model and the morphism of cdgas m' is a quasi-isomorphism.

The reader should skip the following remark on his first reading.

Remark 4.2. 1) In the previous proof, if the composites $m_X \circ \varphi$ and $A_{PL}(f) \circ m_B$ are not strictly equal then the map m' is not well defined. In general, the composites $m_X \circ \varphi$ and $A_{PL}(f) \circ m_B$ are only homotopic and the situation is more complicated: see part 2) of this remark.

2) Let $m_B : \Lambda B \overset{\simeq}{\twoheadrightarrow} A_{PL}(B)$ be a Sullivan model of B. Let $m'_X : \Lambda X' \overset{\simeq}{\twoheadrightarrow} A_{PL}(X)$ be a Sullivan model of X. By the lifting Lemma of Sullivan models [5, Proposition 14.6], there exists a morphism of cdgas $\varphi' : \Lambda B \to \Lambda X'$ such that the following diagram commutes only up to homotopy (in the sense of [6, Section 2.2])

$$\begin{tikzcd}
A_{PL}(B) \arrow[r, "A_{PL}(f)"] & A_{PL}(X) \\
\Lambda B \arrow[u, "m_B", "\simeq"'] \arrow[r, "\varphi'"] & \Lambda X'. \arrow[u, "m'_X", "\simeq"']
\end{tikzcd}$$

In general, this square is not strictly commutative. Let $\Lambda B \hookrightarrow \Lambda B \otimes \Lambda V$ be a relative Sullivan model of $A_{PL}(p) \circ m_B$. Then there exists a commutative diagram of cdgas

$$\begin{tikzcd}
A_{PL}(X) \arrow[r, "A_{PL}(q)"] & A_{PL}(P) \\
\Lambda X \arrow[u, "\simeq"'] \arrow[r] \arrow[d, "\simeq"] & \Lambda X \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V) \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\
\Lambda X' \arrow[r] & \Lambda X' \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V)
\end{tikzcd}$$

Proof of part 2) of Remark 4.2. Let $\Lambda B \overset{\varphi}{\hookrightarrow} \Lambda X \overset{\theta}{\to} \Lambda X'$ be a relative Sullivan model of φ'. Since the composites $m'_X \circ \theta \circ \varphi$ and $A_{PL}(f) \circ m_B$ are homotopic, by the homotopy extension property [6, Proposition 2.22] of the relative Sullivan model $\varphi : \Lambda B \hookrightarrow \Lambda X$, there exists a morphism of cdgas $m_X : \Lambda X \to A_{PL}(X)$ homotopic to $m'_X \circ \theta$ such that $m_X \circ \varphi = A_{PL}(f) \circ m_B$. Therefore using diagram (1), we obtain the following commutative diagram of cdgas:

$$\begin{tikzcd}
A_{PL}(X) \arrow[r, "A_{PL}(q)"] & A_{PL}(P) & A_{PL}(E) \arrow[l, "A_{PL}(g)"'] \\
\Lambda X \arrow[u, "\simeq"', "m_X"] \arrow[r] \arrow[d, "\simeq"', "\theta"] & \Lambda X \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V) \arrow[u, "\simeq"', "m'"] \arrow[d, "\simeq"', "\theta \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V)"] & \Lambda B \otimes \Lambda V \arrow[l] \arrow[u, "\simeq"', "m"] \\
\Lambda X' \arrow[r] & \Lambda X' \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V). &
\end{tikzcd}$$

Here, since θ is a quasi-isomorphism, the pushout morphism $\theta \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V)$ along the relative Sullivan model $\Lambda X \hookrightarrow \Lambda X \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V)$ is also a quasi-isomorphism [5, Lemma 14.2]. $\square$

4.4 Sullivan model of a fibration

Let $p : E \to B$ be a (Serre) fibration with fibre $F := p^{-1}(b_0)$.

$$\begin{tikzcd}
F \arrow[r, "j"] \arrow[d] & E \arrow[d, "p"] \\
b_0 \arrow[r] & B
\end{tikzcd}$$

Taking X to be the point b_0, we can apply the results of the previous section. Let $m_B : (\Lambda V, d) \overset{\simeq}{\to} A_{PL}(B)$ be a Sullivan model of B. Let $(\Lambda V, d) \hookrightarrow (\Lambda V \otimes \Lambda W, d)$ be a relative Sullivan model of $A_{PL}(p) \circ m_B$.

Since $A_{PL}(\{b_0\})$ is equal to $(\mathbf{k}, 0)$, there is a unique morphism of cdgas m' such that the following diagram commutes

$$\begin{tikzcd}
A_{PL}(B) \arrow[r, "A_{PL}(p)"] & A_{PL}(E) \arrow[r, "A_{PL}(j)"] & A_{PL}(F) \\
(\Lambda V, d) \arrow[u, "\simeq"', "m_B"] \arrow[r] & (\Lambda V \otimes \Lambda W, d) \arrow[u, "\simeq"'] \arrow[r] & (k,0) \otimes_{(\Lambda V,d)} (\Lambda V \otimes \Lambda W, d) \arrow[u, "m'"']
\end{tikzcd}$$

Suppose that the base B is a simply connected space and that the total space E is path-connected. Then by the previous section, the morphism of cdga's

$$m' : (k,0) \otimes_{(\Lambda V,d)} (\Lambda V \otimes \Lambda W, d) \cong (\Lambda W, \bar{d}) \xrightarrow{\simeq} A_{PL}(F)$$

is a quasi-isomorphism:

" The cofiber of a relative Sullivan model of a fibration is a Sullivan model of the fiber of the fibration."

Note that the cofiber of a relative Sullivan model is minimal if and only if the relative Sullivan model is minimal.

4.5 Sullivan model of free loop spaces

Let X be a simply-connected space. Consider the commutative diagram of spaces

$$\begin{tikzcd}
X^{S^1} \arrow[r] \arrow[d, "ev"'] & X^I \arrow[d, "(ev_0,ev_1)"'] & X \arrow[l, "\sigma"', "\approx"] \arrow[dl, "\Delta"] \\
X \arrow[r, "\Delta"'] & X \times X &
\end{tikzcd}$$

where the square is a pullback. Here I denotes the closed interval $[0,1]$, ev, ev_0, ev_1 are the evaluation maps and the homotopy equivalence $\sigma : X \overset{\approx}{\to} X^I$ is the inclusion of constant paths. Let $m_X : \Lambda V \overset{\simeq}{\to} A_{PL}(X)$ be a minimal Sullivan model of X. By Proposition 2.8, the multiplication $\mu : \Lambda V \otimes \Lambda V \to \Lambda V$ admits a minimal relative Sullivan model of the form

$$\Lambda V \otimes \Lambda V \hookrightarrow \Lambda V \otimes \Lambda V \otimes \Lambda sV.$$

Since μ is a model of the diagonal (Section 4.2) and since $\Delta = (ev_0, ev_1) \circ \sigma$, we have the commutative rectangle of cdgas

$$\begin{tikzcd}
A_{PL}(X \times X) \arrow[r, "A_{PL}((ev_0,ev_1))"] & A_{PL}(X^I) \arrow[r, "A_{PL}(\sigma)"] & A_{PL}(X) \\
\Lambda V \otimes \Lambda V \arrow[u, "m_{X\times X}", "\simeq"'] \arrow[r] & \Lambda V \otimes \Lambda V \otimes \Lambda sV \arrow[r, "\simeq"'] & \Lambda V \arrow[u, "\simeq", "m_X"']
\end{tikzcd}$$

Since σ is a homotopy equivalence, $S^*(\sigma)$ is a homotopy equivalence of complexes and in particular a quasi-isomorphim. So by Theorem 3.1 and naturality, $A_{PL}(\sigma)$ is also a quasi-isomorphism. Therefore, by the lifting property of relative Sullivan models [5, Proposition 14.6], there exists a morphism of cdgas $\varphi : \Lambda V \otimes \Lambda V \otimes$

$\Lambda sV \to A_{PL}(X^I)$ such that, in the diagram of cdgas

$$\begin{tikzcd} A_{PL}(X\times X) \arrow[r, "A_{PL}((ev_0,ev_1))"] & A_{PL}(X^I) \arrow[r, "A_{PL}(\sigma)", "\simeq"'] & A_{PL}(X) \\ \Lambda V\otimes\Lambda V \arrow[u, "m_{X\times X}", "\simeq"'] \arrow[r] & \Lambda V\otimes\Lambda V\otimes\Lambda sV \arrow[u, dotted, "\varphi", "\simeq"'] \arrow[r, "\simeq"'] & \Lambda V \arrow[u, "\simeq", "m_X"'] \end{tikzcd}$$

the left square commutes exactly and the right square commutes in homology. Therefore φ is also a quasi-isomorphism. This means that

$$\Lambda V\otimes\Lambda V \hookrightarrow \Lambda V\otimes\Lambda V\otimes\Lambda sV.$$

is a relative Sullivan model of the composite

$$\Lambda V\otimes\Lambda V \overset{m_{X\times X}}{\to} A_{PL}(X\times X) \overset{A_{PL}((ev_0,ev_1))}{\longrightarrow} A_{PL}(X^I).$$

Here diagram (1) specializes to the following commutative diagram of cdgas

$$\begin{tikzcd} & \Lambda V\otimes\Lambda V \arrow[r, "\mu"] \arrow[d] & \Lambda V \arrow[r, "m_X", "\simeq"'] \arrow[d] & A_{PL}(X) \arrow[dd, "A_{PL}(ev)"] \\ & \Lambda V\otimes\Lambda V\otimes\Lambda sV \arrow[r] \arrow[dl, "\varphi"', "\simeq"] & \Lambda V\otimes_{\Lambda V\otimes\Lambda V}\Lambda V\otimes\Lambda V\otimes\Lambda sV \arrow[dr, "\simeq"] & \\ A(X^I) \arrow[rrr] & & & A(X^{S^1}) \end{tikzcd} \tag{2}$$

where the rectangle is a pushout. Therefore

$$\Lambda V \hookrightarrow \Lambda V\otimes_{\Lambda V\otimes\Lambda V}(\Lambda V\otimes\Lambda V\otimes\Lambda sV)\cong(\Lambda V\otimes\Lambda sV,\delta)$$

is a minimal relative Sullivan model of $A_{PL}(ev)\circ m_X$.

Corollary 4.3. *Let X be a simply-connected space. Then the free loop space cohomology of $H^*(X^{S^1};\mathbf{k})$ with coefficients in a field $\mathbf{k}$ of characteristic 0 is isomorphic to the Hochschild homology of $A_{PL}(X)$, $HH_*(A_{PL}(X),A_{PL}(X))$.*

Replacing $A_{PL}(X)$ by $A_{DR}(M)$ (Remark 3.2), this Corollary is a theorem of Chen [3, 3.2.3 Theorem] when X is a smooth manifold M.

Proof. The quasi-isomorphism of cdgas $m_X:\Lambda V\overset{\simeq}{\to}A_{PL}(X)$ induces an isomorphism between Hochschild homologies

$$HH_*(m_X,m_X):HH_*(\Lambda V,\Lambda V)\overset{\cong}{\to}HH_*(A_{PL}(X),A_{PL}(X)).$$

By [5, Lemma 14.1], $\Lambda V\otimes\Lambda V\otimes\Lambda sV$ is a semi-free resolution of ΛV as a $\Lambda V\otimes\Lambda V^{op}$-module. Therefore the Hochschild homology $HH_*(\Lambda V,\Lambda V)$ can be defined as the homology of the cdga $(\Lambda V\otimes\Lambda sV,\delta)$. We have just seen above that $H(\Lambda V\otimes\Lambda sV,\delta)$ is isomorphic to the free loop space cohomology $H^*(X^{S^1};\mathbf{k})$. □

We have shown that a Sullivan model of X^{S^1} is of the form $(\Lambda V\otimes\Lambda sV,\delta)$. The following theorem of Vigué-Poirrier and Sullivan gives a precise description of the differential δ.

Theorem 4.4. *([17, Theorem p. 637] or [6, Theorem 5.11]) Let X be a simply connected topological space. Let $(\Lambda V,d)$ be a minimal Sullivan model of X. For all $v\in V$, denote by sv an element of degree $|v|-1$. Let $s:\Lambda V\otimes\Lambda sV\to\Lambda V\otimes\Lambda sV$ be the unique derivation of (upper) degree -1 such that on the generators v, sv, $v\in V$, $s(v)=sv$ and $s(sv)=0$. We have $s\circ s=0$. Then there exists a unique Sullivan model of X^{S^1} of the form $(\Lambda V\otimes\Lambda sV,\delta)$ such that $\delta\circ s+s\circ\delta=0$ on $\Lambda V\otimes\Lambda sV$.*

Remark 4.5. Consider the free loop fibration $\Omega X\hookrightarrow X^{S^1}\overset{ev}{\twoheadrightarrow}X$. Since $(\Lambda V,d)\hookrightarrow(\Lambda V\otimes\Lambda sV,\delta)$ is a minimal relative Sullivan model of $A_{PL}(ev)\circ m_X$, by Section 4.4,

$$\Bbbk\otimes_{(\Lambda V,d)}(\Lambda V\otimes\Lambda sV,\delta)\cong(\Lambda sV,\bar\delta)$$

is a minimal Sullivan model of ΩX. Let $v\in V$. By Theorem 4.4, $\delta(sv)=-s\delta v=-sdv$. Since $dv\in\Lambda^{\geq 2}V$, $\delta(sv)\in\Lambda^{\geq 1}V\otimes\Lambda^1 sV$. Therefore $\bar\delta=0$. Since ΩX is a H-space, this follows also from Theorem 5.3 and from the unicity of minimal Sullivan models (part 1) of Theorem 3.4).

5 Examples of Sullivan models

5.1 Sullivan model of spaces with polynomial cohomology

The following proposition is a straightforward generalisation [5, p. 144] of the Sullivan model of odd-dimensional spheres (see section 1.2).

Proposition 5.1. *Let X be a path connected topological space such that its cohomology $H^*(X;\mathbf{k})$ is a free graded commutative algebra ΛV (for example, polynomial). Then a Sullivan model of X is $(\Lambda V,0)$.*

Example 5.2. Odd-dimensional spheres S^{2n+1}, complex or quartenionic Stiefel manifolds [6, Example 2.40] $V_k(\mathbb{C}^n)$ or $V_k(\mathbb{H}^n)$, classifying spaces BG of simply connected Lie groups [6, Example 2.42], connected Lie groups G as we will see in the following section.

5.2 Sullivan model of an H-space

An *H-space* is a pointed topological space (G,e) equipped with a pointed continuous map $\mu:(G,e)\times(G,e)\to(G,e)$ such that the two pointed maps $g\mapsto\mu(e,g)$ and $g\mapsto\mu(g,e)$ are pointed homotopic to the identity map of (G,e).

Theorem 5.3. *[5, Example 3 p. 143] Let G be a path connected H-space such that $\forall n \in \mathbb{N}$, $H_n(G;\mathbf{k})$ is finite dimensional. Then*
1) its cohomology $H^*(G;\mathbf{k})$ is a free graded commutative algebra ΛV,
2) G has a Sullivan model of the form $(\Lambda V, 0)$, that is with zero differential.

Proof. 1) Let A be $H^*(G;\mathbf{k})$ the cohomology of G. By hypothesis, A is a connected commutative graded Hopf algebra (not necessarily associative). Now the theorem of Hopf-Borel in caracteristic 0 [4, VII.10.16] says that A is a free graded commutative algebra.

2) By Proposition 5.1, 1) and 2) are equivalent. □

Example 5.4. Let G be a path-connected Lie group (or more generally a H-space with finitely generated integral homology). Then G has a Sullivan model of the form $(\Lambda V, 0)$. By Theorem 3.4, V^n and $\pi_n(G) \otimes_{\mathbb{Z}} \mathbf{k}$ have the same dimension for any $n \in \mathbb{N}$. Since $H_*(G;\mathbf{k})$ is of finite (total) dimension, V and therefore $\pi_*(G) \otimes_{\mathbb{Z}} \mathbf{k}$ are concentrated in odd degrees. In fact, more generally [2, Theorem 6.11], $\pi_2(G) = \{0\}$. Note, however that $\pi_4(S^3) = \mathbb{Z}/2\mathbb{Z} \neq \{0\}$.

5.3 Sullivan model of projective spaces

Consider the complex projective space $\mathbb{CP}^n$, $n \geq 1$. The construction of the Sullivan model of $\mathbb{CP}^n$ is similar to the construction of the Sullivan model of $S^2 = \mathbb{CP}^1$ done in section 1.2:

The cohomology algebra $H^*(A_{PL}(\mathbb{CP}^n)) \cong H^*(\mathbb{CP}^n)$ is the truncated polynomial algebra $\frac{\mathbf{k}[x]}{x^{n+1}=0}$ where x is an element of degree 2. Let v be a cycle of $A_{PL}(\mathbb{CP}^n)$ representing $x := [v]$. The inclusion of complexes $(\mathbf{k}v, 0) \hookrightarrow A_{PL}(\mathbb{CP}^n)$ extends to a unique morphism of cdgas $m : (\Lambda v, 0) \to A_{PL}(\mathbb{CP}^n)$(Property 1.6). Since $[v^{n+1}] = x^{n+1} = 0$, there exists an element $\psi \in A_{PL}(\mathbb{CP}^n)$ of degree $2n+1$ such that $d\psi = v^{n+1}$. Let w denote another element of degree $2n+1$. Let d be the unique derivation of $\Lambda(v, w)$ such that $d(v) = 0$ and $d(w) = v^{n+1}$. The unique morphism of graded algebras $m : (\Lambda(v, w), d) \to A_{PL}(\mathbb{CP}^n)$ such that $m(v) = v$ and $m(w) = \psi$, is a morphism of cdgas. In homology, $H(m)$ sends 1, $[v]$, …, $[v^n]$ to 1, x, …, x^n. Therefore m is a quasi-isomorphism.

More generally, let X be a simply connected space such that $H^*(X)$ is a truncated polynomial algebra $\frac{\mathbf{k}[x]}{x^{n+1}=0}$ where $n \geq 1$ and x is an element of even degree $d \geq 2$. Then the Sullivan model of X is $(\Lambda(v, w), d)$ where v is an element of degree d, w is an element of degree $d(n+1) - 1$, $d(v) = 0$ and $d(w) = v^{n+1}$.

5.4 Free loop space cohomology for even-dimensional spheres and projective spaces

In this section, we compute the free loop space cohomology of any simply connected space X whose cohomology is a truncated polynomial algebra $\frac{\mathbf{k}[x]}{x^{n+1}=0}$ where $n \geq 1$ and x is an element of even degree $d \geq 2$.

Mainly, this is the even-dimensional sphere S^d ($n = 1$), the complex projective space $\mathbb{CP}^n$ ($d = 2$), the quaternionic projective space $\mathbb{HP}^n$ ($d = 4$) and the Cayley plane $\mathbb{OP}^2$ ($n = 2$ and $d = 8$).

In the previous section, we have seen that the minimal Sullivan model of X is $(\Lambda(v,w), d(v) = 0, d(w) = v^{n+1})$ where v is an element of degree d and w is an element of degree $d(n+1)-1$. By the constructive proof of Proposition 2.8, the multiplication μ of this minimal Sullivan model $(\Lambda(v,w), d)$ admits the relative Sullivan model $(\Lambda(v,w)\otimes\Lambda(v,w)\otimes\Lambda(sv,sw), D)$ where

$$D(1\otimes 1\otimes sv) = v\otimes 1\otimes 1 - 1\otimes v\otimes 1 \text{ and}$$

$$D(1\otimes 1\otimes sw) = w\otimes 1\otimes 1 - 1\otimes w\otimes 1 - \sum_{i=0}^{n} v^i\otimes v^{n-i}\otimes sv.$$

Therefore, by taking the pushout along μ of this relative Sullivan model (diagram (2)), or simply by applying Theorem 4.4, a relative Sullivan model of $A_{PL}(ev)\circ m_X$ is given by the inclusion of cdgas $(\Lambda(v,w), d) \hookrightarrow (\Lambda(v,w,sv,sw),\delta)$ where $\delta(sv) = -sd(v) = 0$ and $\delta(sw) = -s(v^{n+1}) = -(n+1)v^n sv$. Consider the pushout square of cdgas

$$\begin{tikzcd}
(\Lambda(v,w), d) \arrow[r] \arrow[d, "\theta", "\simeq"'] & (\Lambda(v,w,sv,sw),\delta) \arrow[d, "\theta\otimes_{\Lambda(v,w)}\Lambda(sv,sw)", "\simeq"'] \\
\left(\frac{\mathbf{k}[v]}{v^{n+1}=0}, 0\right) \arrow[r] & \left(\frac{\mathbf{k}[v]}{v^{n+1}=0}\otimes\Lambda(sv,sw),\bar{\delta}\right).
\end{tikzcd}$$

Here, since θ is a quasi-isomorphism, the pushout morphism $\theta\otimes_{\Lambda(v,w)}\Lambda(sv,sw)$ along the relative Sullivan model $\Lambda(v,w)\hookrightarrow\Lambda(v,w,sv,sw)$ is also a quasi-isomorphism [5, Lemma 14.2]. Therefore, $H^*(X^{S^1};\mathbf{k})$ is the graded vector space

$$\mathbb{k}\oplus\bigoplus_{1\le p\le n,\ i\in\mathbb{N}}\mathbb{k}v^p(sw)^i\oplus\bigoplus_{0\le p\le n-1,\ i\in\mathbb{N}}\mathbb{k}v^p sv(sw)^i.$$

(In [11, Section 8], the author extends these rational computations over any commutative ring.) Since for all $i\in\mathbb{N}$, the degree of $v(sw)^{i+1}$ is strictly greater than the degree of $v^n(sw)^i$, the generators 1, $v^p(sw)^i$, $1\le p\le n$, $i\in\mathbb{N}$, have all distinct (even) degrees. Since for all $i\in\mathbb{N}$, the degree of $sv(sw)^{i+1}$ is strictly greater than the degree of $v^{n-1}sv(sw)^i$, the generators $v^p sv(sw)^i$, $0\le p\le n-1$, $i\in\mathbb{N}$, have also distinct (odd) degrees. Therefore, for all $p\in\mathbb{N}$, Dim $H^p(X^{S^1};\mathbf{k})\le 1$.

At the end of section 2.2, we have shown the same inequalities when X is an odd-dimensional sphere, or more generally for a simply-connected space X whose cohomology $H^*(X;\mathbf{k})$ is an exterior algebra Λx on an odd degree generator x. Since every finite dimensional graded commutative algebra generated by a single element x is either Λx or $\frac{\mathbf{k}[x]}{x^{n+1}=0}$, we have shown the following proposition:

Proposition 5.5. *Let X be a simply connected topological space such that its cohomology $H^*(X;\mathbf{k})$ is generated by a single element and is finite dimensional. Then the sequence of Betti numbers of the free loop space on X, $b_n :=$ dim $H^n(X^{S^1};\mathbf{k})$*

is bounded.

The goal of the following section will be to prove the converse of this proposition.

6 Vigué-Poirrier-Sullivan theorem on closed geodesics

The goal of this section is to prove (See section 6.4) the following theorem due to Vigué-Poirrier and Sullivan.

6.1 Statement of Vigué-Poirrier-Sullivan theorem and of its generalisations

Theorem 6.1. *([17, Theorem p. 637] or [6, Proposition 5.14]) Let M be a simply connected topological space such that the rational cohomology of M, $H^*(M;\mathbb{Q})$ is of finite (total) dimension (in particular, vanishes in higher degrees).*

If the cohomology algebra $H^*(M;\mathbb{Q})$ requires at least two generators then the sequence of Betti numbers of the free loop space on M, $b_n := dim\ H^n(M^{S^1};\mathbb{Q})$ is unbounded.

Example 6.2. (Betti numbers of $(S^3\times S^3)^{S^1}$ over $\mathbb{Q}$)

Let V and W be two graded vector spaces such $\forall n\in\mathbb{N}$, V^n and W^n are finite dimensional. We denote by

$$P_V(z) := \sum_{n=0}^{+\infty}(\text{Dim } V^n)z^n$$

the sum of the *Poincaré serie* of V. If V is the cohomology of a space X, we denote $P_{H^*(X)}(z)$ simply by $P_X(z)$. Note that $P_{V\otimes W}(z)$ is the product $P_V(z)P_W(z)$. We saw at the end of section 2.2 that $H^*((S^3)^{S^1};\mathbb{Q})\cong \Lambda v\otimes\Lambda sv$ where v is an element of degree 3. Therefore

$$P_{(S^3)^{S^1}}(z) = (1+z^3)\sum_{n=0}^{+\infty} z^{2n} = \frac{1+z^3}{1-z^2}.$$

Since the free loops on a product is the product of the free loops

$$H^*((S^3\times S^3)^{S^1}) \cong H^*((S^3)^{S^1})\otimes H^*((S^3)^{S^1}).$$

Therefore, since $\dfrac{1}{1-z^2} = \sum_{n=0}^{+\infty}(n+1)z^{2n}$,

$$P_{(S^3\times S^3)^{S^1}}(z) = \left(\frac{1+z^3}{1-z^2}\right)^2 = 1 + 2z^2 + \sum_{n=3}^{+\infty}(n-1)z^n.$$

So the Betti numbers over $\mathbb{Q}$ of the free loop space on $S^3\times S^3$, $b_n :=$ Dim $H^n((S^3\times S^3)^{S^1};\mathbb{Q})$ are equal to $n-1$ if $n\geq 3$. In particular, they are unbounded.

Conjecture 6.3. *The theorem of Vigué-Poirrier and Sullivan holds replacing $\mathbb{Q}$ by any field $\mathbb{F}$.*

Example 6.4. (Betti numbers of $(S^3\times S^3)^{S^1}$ over $\mathbb{F}$)

The calculation of Example 6.2 over $\mathbb{Q}$ can be extended over any field $\mathbb{F}$ as follows: Since S^3 is a topological group, the map $\Omega S^3\times S^3\rightarrow (S^3)^{S^1}$, sending (w,g) to the free loop $t\mapsto w(t)g$, is a homeomorphism. Using Serre spectral sequence ([13, Proposition 17] or[14, Chap 9. Sect 7. Lemma 3]) or Bott-Samelson theorem ([12, Corollary 7.3.3] or [9, Appendix 2 Theorem 1.4]), the cohomology of the pointed loops on S^3, $H^*(\Omega S^3)$ is again isomorphic (as graded vector spaces only!) to the polynomial algebra Λsv where sv is of degree 2. Therefore exactly as over $\mathbb{Q}$, $H^*((S^3)^{S^1};\mathbb{F})\cong\Lambda v\otimes\Lambda sv$ where v is an element of degree 3. Now the same proof as in Example 6.2 shows that the Betti numbers over $\mathbb{F}$ of the free loop space on $S^3\times S^3$, $b_n :=$ Dim $H^n((S^3\times S^3)^{S^1};\mathbb{F})$ are again equal to $n-1$ if $n\geq 3$.

In fact, the theorem of Vigué-Poirrier and Sullivan is completely algebraic:

Theorem 6.5. *([17] when $\mathbb{F}=\mathbb{Q}$, [7, Theorem III p. 315] over any field $\mathbb{F}$) Let $\mathbb{F}$ be a field. Let A be a cdga such that $H^{<0}(A)=0$, $H^0(A)=\mathbb{F}$ and $H^*(A)$ is of finite (total) dimension. If the algebra $H^*(A)$ requires at least two generators then the sequence of dimensions of the Hochschild homology of A, $b_n :=$ dim $HH_{-n}(A,A)$ is unbounded.*

Generalising Chen's theorem (Corollary 4.3) over any field $\mathbb{F}$, Jones theorem [10] gives the isomorphisms of vector spaces

$$H^n(X^{S^1};\mathbb{F})\cong HH_{-n}(S^*(X;\mathbb{F}),S^*(X;\mathbb{F})),\quad n\in\mathbb{Z}$$

between the free loop space cohomology of X and the Hochschild homology of the algebra of singular cochains on X. But since the algebra of singular cochains $S^*(X;\mathbb{F})$ is not commutative, Conjecture 6.3 does not follow from Theorem 6.5.

6.2 A first result of Sullivan

In this section, we start by a first result of Sullivan whose simple proof illustrates the technics used in the proof of Vigué-Poirrier-Sullivan theorem.

Theorem 6.6. *[15] Let X be a simply-connected space such that $H^*(X;\mathbb{Q})$ is not concentrated in degree 0 and $H^n(X;\mathbb{Q})$ is null for n large enough. Then on the contrary, $H^n(X^{S^1};\mathbb{Q}) \neq 0$ for an infinite set of integers n.*

Proof. Let $(\Lambda V, d)$ be a minimal Sullivan model of X. Suppose that V is concentrated in even degree. Then $d = 0$. Therefore $H^*(\Lambda V, d) = \Lambda V$ is either concentrated in degree 0 or is not null for an infinite sequence of degrees. By hypothesis, we have excluded theses two cases. Therefore dim $V^{odd} \geq 1$.

Let x_1, x_2, $\dots$, x_m, y, x_{m+1}, ... be a basis of V ordered by degree where y denotes the first generator of odd degree ($m \geq 0$). For all $1 \leq i \leq m$, $dx_i \in \Lambda x_{<i}$. But dx_i is of odd degree and $\Lambda x_{<i}$ is concentrated in even degre. So $dx_i = 0$. Since $dy \in \Lambda x_{\leq m}$, dy is equal to a polynomial $P(x_1, \dots, x_m)$ which belongs to $\Lambda^{\geq 2}(x_1, \dots, x_m)$.

Consider $(\Lambda V \otimes \Lambda sV, \delta)$, the Sullivan model of X^{S^1}, given by Theorem 4.4. We have $\forall 1 \leq i \leq m$, $\delta(sx_i) = -sdx_i = 0$ and $\delta(sy) = -sdy \in \Lambda^{\geq 1}(x_1, \dots, x_m) \otimes \Lambda^1(sx_1, \dots, sx_m)$. Therefore, since sx_1,…,sx_m are all of odd degree, $\forall p \geq 0$,

$$\delta(sx_1 \dots sx_m (sy)^p) = \pm sx_1 \dots sx_m p \delta(sy)(sy)^{p-1} = 0.$$

For all $p \geq 0$, the cocycle $sx_1 \dots sx_m (sy)^p$ gives a non trivial cohomology class in $H^*(X^{S^1};\mathbb{Q})$, since by Remark 4.5, the image of this cohomology class in $H^*(\Omega X;\mathbb{Q}) \cong \Lambda V$ is different from 0. □

6.3 Dimension of $V^{odd} \geq 2$

In this section, we show the following proposition:

Proposition 6.7. *Let X be a simply connected space such that $H^*(X;\mathbb{Q})$ is of finite (total) dimension and requires at least two generators. Let $(\Lambda V, d)$ be the minimal Sullivan model of X. Then dim $V^{odd} \geq 2$.*

Property 6.8. (Koszul complexes) Let A be a graded algebra. Let z be a central element of even degree of A which is not a divisor of zero. Then we have a quasi-isomorphism of dgas

$$(A \otimes \Lambda sz, d) \overset{\simeq}{\twoheadrightarrow} A/z.A \quad a \otimes 1 \mapsto a, a \otimes sz \mapsto 0,$$

where $d(a \otimes 1) = 0$ and $d(a \otimes sz) = (-1)^{|a|} az$ for all $a \in A$.

Proof of Proposition 6.7 (following $(2) \Rightarrow (3)$ *of p. 214 of [6]).* As we saw in the proof of Theorem 6.6, there is at least one generator y of odd degree, that is dim $V^{odd} \geq 1$. Suppose that there is only one. Let x_1, x_2, $\dots$, x_m, y, x_{m+1},… be a basis of V ordered by degree ($m \geq 0$).

First case: $dy = 0$. If $m \geq 1$, $dx_1 = 0$. If $m = 0$, $dx_1 \in \Lambda^{\geq 2}(y) = \{0\}$ and therefore again $dx_1 = 0$. Suppose that for $n \geq 1$, x_1^n is a coboundary. Then $x_1^n = d(yP(x_1, \dots)) = yd(P(x_1, \dots))$ where $P(x_1, \dots)$ is a polynomial in the x_i's. But this is impossible since x_1^n does not belong to the ideal generated by y.

Therefore for all $n \geq 1$, x_1^n gives a non trivial cohomology class in $H^*(X)$. But $H^*(X)$ is finite dimensional.

Second case: $dy \neq 0$. In particular $m \geq 1$. Since dy is a non zero polynomial, dy is not a zero divisor, so by Property 6.8, we have a quasi-isomorphism of cdgas

$$\Lambda(x_1, \dots, x_m, y) \overset{\simeq}{\twoheadrightarrow} \Lambda(x_1, \dots, x_m)/(dy).$$

Consider the push out in the category of cdgas

$$\begin{tikzcd}
\Lambda(x_1, \dots, x_m, y) \arrow[r] \arrow[d, "\simeq"'] & \Lambda(x_1, \dots, x_m, y, x_{m+1}, \dots), d \arrow[d] \\
\Lambda(x_1, \dots, x_m)/(dy) \arrow[r] & \Lambda(x_1, \dots, x_m)/(dy) \otimes \Lambda(x_{m+1}, \dots), \bar{d}
\end{tikzcd}$$

Since $\Lambda(x_1, \dots, x_m)/(dy) \otimes \Lambda(x_{m+1}, \dots)$ is concentrated in even degrees, $\bar{d} = 0$. Since the top arrow is a Sullivan relative model and the left arrow is a quasi-isomorphism, the right arrow is also a quasi-isomorphism ([5, Lemma 14.2], or more generally the category of cdgas over $\mathbb{Q}$ is a Quillen model category). Therefore the algebra $H^*(X)$ is isomorphic to $\Lambda(x_1, \dots, x_m)/(dy) \otimes \Lambda(x_{m+1}, \dots)$. If $m \geq 2$, $\Lambda(x_1, \dots, x_m)/(dy)$ and so $H^*(X)$ is infinite dimensional. If $m = 1$, since $\Lambda x_1/(dy)$ is generated by only one generator, we must have another generator x_2. But $\Lambda(x_1)/(dy) \otimes \Lambda(x_2, \dots)$ is also infinite dimensional. □

6.4 Proof of Vigué-Poirrier-Sullivan theorem

Lemma 6.9. *[17, Proposition 4] Let A be a dga over any field such that the multiplication by a cocycle x of any degre $A \to A$, $a \mapsto xa$ is injective (Our example will be $A = (\Lambda V, d)$ and x a non-zero element of V of even degree such that $dx = 0$). If the Betti numbers $b_n = \dim H^n(A)$ of A are bounded then the Betti numbers $b_n = \dim H^n(A/xA)$ of A/xA are also bounded.*

Proof. Since $H^n(xA) \cong H^{n-|x|}(A)$, the short exact sequence of complexes

$$0 \to xA \to A \to A/xA \to 0$$

gives the long exact sequence in homology

$$\dots \to H^n(A) \to H^n(A/xA) \to H^{n+1-|x|}(A) \to \dots$$

Therefore $\dim H^n(A/xA) \leq \dim H^n(A) + \dim H^{n+1-|x|}(A)$ □

Proof of Vigué-Poirrier-Sullivan theorem (Theorem 6.1). Let $(\Lambda V, d)$ be the minimal Sullivan model of X. Let $(\Lambda V \otimes \Lambda sV, \delta)$ be the Sullivan model of X^{S^1} given by Theorem 4.4. From Proposition 6.7, we know that $\dim V^{odd} \geq 2$. Let x_1, x_2, $\dots$, x_m, y, $x_{m+1}, \dots$, x_n, $z = x_{n+1}$, $\dots$ be a basis of V ordered by degrees where $x_1, \dots$, x_n are of even degrees and y, z are of odd degrees. Consider the commutative diagram of cdgas where the three rectangles are push outs

$$\begin{tikzcd}
\Lambda(x_1,\dots,x_n) \arrow[r] \arrow[d] & (\Lambda V,d) \arrow[r] \arrow[d] & (\Lambda V\otimes\Lambda sV,\delta) \arrow[d] \\
\mathbb{Q} \arrow[r] & \Lambda(y,z,\dots) \arrow[r] \arrow[d] & (\Lambda(y,z,\dots)\otimes\Lambda sV,\bar{\delta}) \arrow[d] \\
 & \mathbb{Q} \arrow[r] & (\Lambda sV,0)
\end{tikzcd}$$

Note that by Remark 4.5, the differential on ΛsV is 0.

For all $1\leq j\leq n+1$,

$$\delta x_j = dx_j\in\Lambda^{\geq 2}(x_{<j},y)\subset\Lambda^{\geq 1}(x_{<j})\otimes\Lambda y.$$

Therefore

$$\delta(sx_j)=-s\delta x_j\in\Lambda x_{<j}\otimes\Lambda^1 sx_{<j}\otimes\Lambda y+\Lambda^{\geq 1}(x_{<j})\otimes\Lambda^1 sy.$$

Since $(sx_1)^2=\dots=(sx_{j-1})^2=0$, the product

$$sx_1\dots sx_{j-1}\delta(sx_j)\in\Lambda^{\geq 1}(x_{<j})\otimes\Lambda^1 sy.$$

So $\forall 1\leq j\leq n+1$, $sx_1\dots sx_{j-1}\bar{\delta}(sx_j)=0$. In particular $sx_1\dots sx_n\bar{\delta}(sz)=0$. Similarly, since $dy\in\Lambda^{\geq 2}x_{\leq m}$, $sx_1\dots sx_m\delta(sy)=0$ and so $sx_1\dots sx_n\bar{\delta}(sy)=0$. By induction, $\forall 1\leq j\leq n$, $\bar{\delta}(sx_1\dots sx_j)=0$. In particular, $\bar{\delta}(sx_1\dots sx_n)=0$. So finally, for all $p\geq 0$ and all $q\geq 0$, $\bar{\delta}(sx_1\dots sx_n(sy)^p(sz)^q)=0$. The cocycles $sx_1\dots sx_n(sy)^p(sz)^q$, $p\geq 0$, $q\geq 0$, give linearly independent cohomology classes in $H^*(\Lambda(y,z,\dots)\otimes\Lambda sV,\bar{\delta})$ since their images in $(\Lambda sV,0)$ are linearly independent.

For all $k\geq 0$, there is at least $k+1$ elements of the form $sx_1\dots sx_n(sy)^p(sz)^q$ in degree $|sx_1|+\dots+|sx_n|+k\cdot\mathrm{lcm}(|sy|,|sz|)$ (just take $p=i\cdot\mathrm{lcm}(|sy|,|sz|)/|sy|$ and $q=(k-i)\mathrm{lcm}(|sy|,|sz|)/|sz|$ for i between 0 and k). Therefore the Betti numbers of $H^*(\Lambda(y,z,\dots)\otimes\Lambda sV,\bar{\delta})$ are unbounded.

Suppose that the Betti numbers of $(\Lambda V\otimes\Lambda sV,\delta)$ are bounded. Then by Lemma 6.9 applied to $A=(\Lambda V\otimes\Lambda sV,\delta)$ and $x=x_1$, the Betti numbers of the quotient cdga $(\Lambda(x_2,\dots)\otimes\Lambda sV,\bar{\delta})$ are bounded. By continuing to apply Lemma 6.9 to x_2, x_3, $\dots$, x_n, we obtain that the Betti numbers of the quotient cdga $(\Lambda(y,z,\dots)\otimes\Lambda sV,\bar{\delta}$ are bounded. But we saw just above that they are unbounded. □

7 Further readings

In this last section, we suggest some further readings that we find appropriate for the student.

In [1, Chapter 19], one can find a very short and gentle introduction to rational homotopy that the reader should compare to our introduction.

In this introduction, we have tried to explain that rational homotopy is a functor which transforms homotopy pullbacks of spaces into homotopy pushouts of cdgas. Therefore after our introduction, we advise the reader to look at [8], a more advanced introduction to rational homotopy, which explains the model category of cdgas.

The canonical reference for rational homotopy [5] is highly readable.

In the recent book [6], you will find many geometric applications of rational homotopy. The proof of Vigué-Poirrier-Sullivan theorem we give here, follows more or less the proof given in [6].

We also like [16] recently reprinted because it is the only book where you can find the Quillen model of a space: a differential graded Lie algebra representing its rational homotopy type (instead of a commutative differential graded algebra as the Sullivan model).

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